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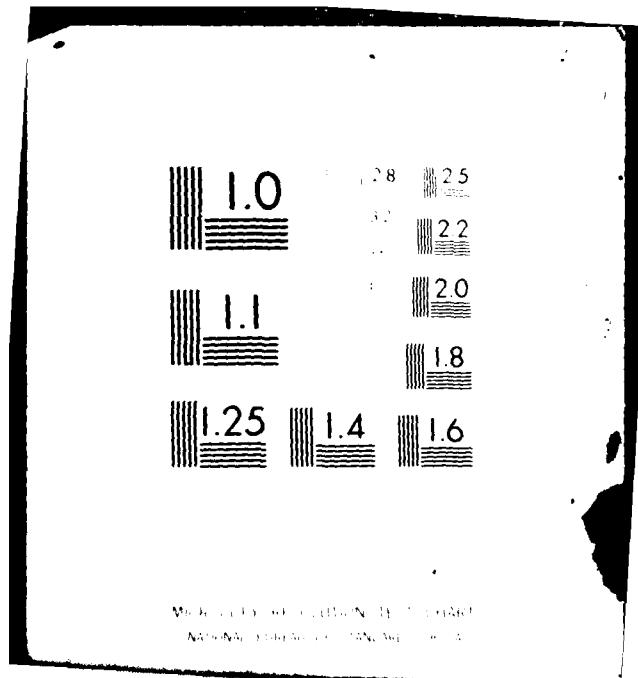
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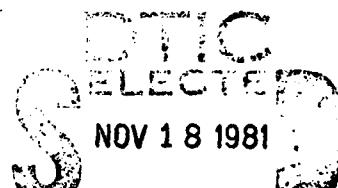
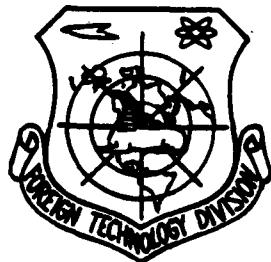
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TABLE OF CONTENTS

Lifting Surface Theory for Wings in Low Frequency Small Amplitude Yawing and Side Slipping Oscillating Motions at Low Speeds, by Huang Mingke.....	1
Simplified Navier-Stokes Equations and Their Mathematical Properties, by Wang Ruquan, Liu Xue Zong, Jiao Luquong, Gao Zhi.....	16
Equations of Flight Mechanics Considering Atmospheric Turbulence, by Jiang Yu-xian.....	28

LIFTING SURFACE THEORY FOR WINGS IN LOW FREQUENCY SMALL AMPLITUDE  
YAWING AND SIDE SLIPPING OSCILLATING MOTIONS AT LOW SPEEDS

Huang Mingke  
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ABSTRACT

In this paper, the mathematical problem of flow at low speeds past a yawing oscillating wing is set up by using a coordinate system fixed in the wing. The snake-like tail vortices behind the wing are investigated in detail. In the case of low frequency oscillations, numerical solution is obtained by the non-steady vortex-lattice method which can be used to calculate the rolling moment aerodynamic derivatives of the wing due to yaw and side slip. Some of the computed results are compared with the experimental results.

I. POSING OF THE MATHEMATICAL PROBLEM OF YAWING

The aerodynamic derivatives caused by yaw and side slip are important primary data for investigating the lateral stability of an airplane. These derivatives are mainly affected by the wing. We take a coordinate system  $oxz$  fixed with the wing as in Figure 1.  $oy$  axis points upward.  $V_\infty$  is the forward speed of the wing.  $\alpha$  and  $\psi$  are the angle of attack and the upper reflex angle respectively. They are both small quantities. We express various quantities in dimensionless form with  $b$ ,  $V_\infty$ ,  $b/V_\infty$  as the characteristic length, characteristic speed and characteristic time. Without losing any generality, we shall only consider the yaw oscillation of the wing with angular velocity  $\omega_y$  around the

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y axis

$$\omega_y = \omega_{y,\max} \cos \omega t$$

where  $t$  is the dimensionless time,  $\omega$  is the circular frequency. We shall let  $\omega_{y,\max} \ll 1$ . Since the addition of yaw oscillation is equivalent to shifting the center of rotation, it is not necessary to specifically study the yaw.

For low speed in compressible flow, corresponding to the small perturbation of the moving coordinate system, the potential flow equation of the absolute motion can be simplified as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (1)$$

in which  $\phi$  is the perturbation velocity potential. If only the first order terms of  $\alpha$  and  $\psi$  are retained, then the integral of the dynamic equation of the small perturbation is simplified as

$$\frac{p - p_\infty}{\rho V_\infty^2} = -\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial y} s\omega_y + \frac{\partial \phi}{\partial z} z\omega_y \quad (2)$$

where  $p$  and  $\rho$  are respectively the pressure and density. The subscript  $\infty$  indicates the condition of free flow.

The boundary conditions on the  $xoz$  plane are as follows:

In the region occupied by the wing:

$$\left( \frac{\partial \phi}{\partial y} \right)_{\text{wing}} = -\sigma - \text{sign}(s)\omega_y x\phi \quad (3)$$

In the wake region, the pressure is continuous so that Equation (2) must be zero. If only the first order term of  $\omega_{y,\max}$  is kept, then the equation is simplified to

$$\left( -\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial y} s\omega_y \right)_{\text{wake}} = 0 \quad (4)$$

Here we have already taken  $(\partial \phi / \partial x)_{\infty} = O(\omega_{y,\max})$  into account.

Outside the wing and wake regions:

$$(\phi)_{\text{out}} = 0 \quad (5)$$

At infinity, the perturbation is zero.

Therefore, the problem of yaw oscillation in flow past wing at low speed is reduced to the solution of Equation (1) with boundary conditions (3) through (5) to be satisfied. Compared with the problem of no yaw, the only difference is in the boundary conditions of the wake region.

## II. THE INSTANTANEOUS SHAPE OF THE WAKE VORTEX LINE IN THE MOVING COORDINATE SYSTEM

Consider the first order linear partial differential Equation (4). Take the differential  $(dx, dz, dt)$  in the  $(x, z, t)$  space so that

$$\frac{dx}{-1} = \frac{dt}{-1} = \frac{dz}{z\omega}, \quad (6)$$

Then Equation (4) means that  $d\Phi = 0$  along this differential element. Hence, it is the linear element of  $\Phi = \text{const}$  in the  $x, z, t$  space. Integrating Equation (6) with the initial conditions  $x_0, z_0, t_0$ , we get

$$\begin{cases} x = x_0 + (t - t_0) \\ z = z_0 + \frac{\omega_{y,\max} x_0 \sin \omega t_0}{\omega} + \frac{\omega_{y,\max} \cos \omega t_0}{\omega^2} - \frac{\omega_{y,\max} x \sin \omega t}{\omega} - \frac{\omega_{y,\max} \cos \omega t}{\omega^2} \end{cases} \quad (7)$$

Let  $x_0, z_0$  represent a point on the rear edge of the wing, then point  $x, z$  is the position reached at time  $t$  by the vortex dragged out from this edge point at time  $t_0$ . Eliminating  $t_0$  from the equations in (7), we get the instantaneous equations for the curve at time  $t$  of the snake-like wake vortex line which starts from the rear edge point  $(x_0, z_0)$ . It can also be seen that the instantaneous shapes of the snake-like wake vortex lines from symmetric points on the rear edges of the left and right wings (equal  $x_0$ ) are identical.

## III. INDUCED VELOCITY CAUSED BY THE SNAKE-LIKE WAKE VORTEX LINE

As shown in Figure 1, we shall calculate the induced velocity  $v_y$  produced by a snake-like wake vortex line at a point

$x_1, z_1$  on the  $y = 0$  plane at time  $t$ . Assuming that the wake vortex circulation  $\Gamma$  does not change with time, then by using the Biot-Savart equation, we get

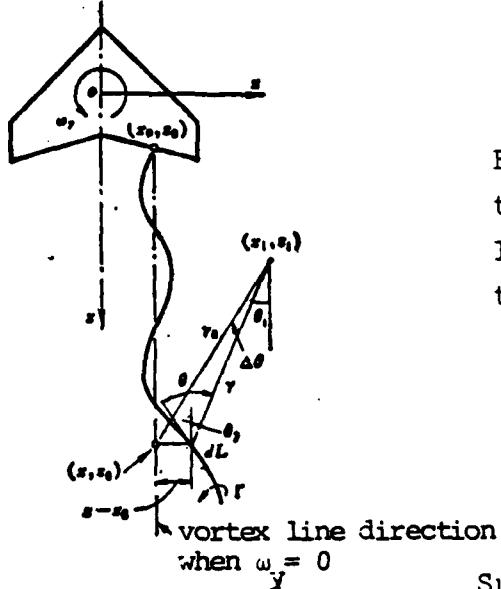


Figure 1

For symbols used, see Figure 1. In the above  $\theta = \theta_1 - \Delta\theta + \theta_0$ ,  $\Delta\theta = O(\omega_{y,\max})$ . If only the first order terms are kept, then

$$\begin{aligned}\sin \theta dL &= (\sin \theta_1 - \cos \theta_1 \Delta\theta) dx \\ &\quad + (\cos \theta_1 + \sin \theta_1 \Delta\theta) dz + O(\omega_{y,\max}^2) \\ \frac{1}{r^2} &= \frac{1}{r_0^2} \left[ 1 + \frac{2 \sin \theta_1}{r_0} (z - z_0) \right] + O(\omega_{y,\max}^2) \\ \Delta\theta &= \frac{(z - z_0) \cos \theta_1}{r_0} + O(\omega_{y,\max}^2)\end{aligned}$$

Substituting into Equation (8), we get

$$\begin{aligned}v_y &= (v_y)_{y=0} - \frac{\Gamma}{4\pi} \left\{ \int_{z_0}^{\infty} \frac{(1 - 3 \sin^2 \theta_1)(z - z_0)}{r_0^3} dx \right. \\ &\quad \left. - \int_{z=z_0}^{\infty} \frac{\cos \theta_1}{r_0^2} dz \right\} + O(\omega_{y,\max}^2) \quad (9)\end{aligned}$$

Substituting the instantaneous wake vortex line equation in the above and simplifying, the two integrals in Equation (9) change respectively to:

$$\begin{aligned}\int_{z_0}^{\infty} \frac{(1 - 3 \sin^2 \theta_1)(z - z_0)}{r_0^3} dx &= \int_{z_0}^{\infty} \frac{1 - 3 \sin^2 \theta_1}{r_0^3} \left\{ \omega_y \left[ -\frac{z_0}{\omega} \sin \omega(x - z_0) \right. \right. \\ &\quad \left. \left. + \frac{1}{\omega^2} \cos \omega(x - z_0) - \frac{1}{\omega^2} \right] + \dot{\omega}_y \left[ -\frac{z_0}{\omega^2} \cos \omega(x - z_0) \right. \right. \\ &\quad \left. \left. - \frac{1}{\omega^3} \sin \omega(x - z_0) + \frac{x}{\omega^3} \right] \right\} dx \quad (10)\end{aligned}$$

$$\begin{aligned}\int_{z=z_0}^{\infty} \frac{\cos \theta_1}{r_0^2} dz &= \int_{z_0}^{\infty} \frac{\cos \theta_1}{r_0^2} \left\{ \dot{\omega}_y \left[ -x_0 \cos \omega(x - z_0) - \frac{1}{\omega} \sin \omega(x - z_0) \right] \right. \\ &\quad \left. + \dot{\omega}_y \left[ \frac{z_0}{\omega} \sin \omega(x - z_0) - \frac{1}{\omega^2} \cos \omega(x - z_0) + \frac{1}{\omega^2} \right] \right\} dx \quad (11)\end{aligned}$$

where  $\dot{\omega}_y = d\omega_y/dt = -\omega_{max}\omega \sin \omega t$ .

The two integrals above apparently both converge but it is difficult to find their analytic forms. The interesting case in airplane stability computation is low frequency oscillation but when  $\omega \rightarrow 0$ , both integrals above are divergent. Luckily, we can separate out the odd terms that make the integrals diverge and discover that when the induced velocity caused by the snake-like wake vortices (with equal circulation but opposite direction) at symmetric points on the rear edges of the left and right wings is computed, these odd terms cancel out one another and the result is still convergent. The derivation follows below.

Take  $x_p$  very large, eventually to approach infinity, then

$$\int_{x_0}^{\infty} - \int_{x_0}^{x_p+x_0} + \int_{x_p+x_0}^{\infty}$$

Consider first integral (10):

In the interval  $(x_0, x_p + x_0)$ , when  $\omega \rightarrow 0$ , in Equation (10)

$$\begin{aligned} & -\frac{x_0}{\omega} \sin \omega(x - x_0) + \frac{1}{\omega^2} \cos \omega(x - x_0) - \frac{1}{\omega^3} \\ & = -(x - x_0)x_0 - \frac{(x - x_0)^2}{2} + O(\omega^2) \\ & -\frac{x_0}{\omega^2} \cos \omega(x - x_0) - \frac{1}{\omega^3} \sin \omega(x - x_0) + \frac{x_0}{\omega^3} \\ & = \frac{(x - x_0)^3}{6} + \frac{x_0(x - x_0)^2}{3} + O(\omega^3) \end{aligned}$$

In the interval  $(x_p + x_0, \infty)$ , since  $x_p \gg 1$ , therefore,

$$\Delta \theta_1 \approx 0, r_1 \approx (x - x_0)^2$$

Hence

$$\int_{x_0+x_0}^{\infty} \frac{(1 - 3\sin^2\theta_1)(x - x_0)dx}{r_0^3} \approx \int_{x_0+x_0}^{\infty} \frac{(x - x_0)dx}{(x - x_1)^2}$$

$$= f(x_p, x_0, x_1, \omega)$$

This integral can be easily integrated. When  $\omega \rightarrow 0$ , it has logarithmic singularity. Since after such integration it will automatically cancel out it is not necessary to integrate the explicit expression. Equation (11) is simplified by the same method. When  $\omega \rightarrow 0$ , Equation (9) simplifies as:

$$v_s = (v_s)_{\omega=0} - \frac{1}{4\pi} \left\{ \omega_s \left[ -x_0 \int_{x_0}^{x_0+x_0} \frac{(1 - 3\sin^2\theta_1)(x - x_0)dx}{r_0^3} \right. \right. \\ \left. \left. - \frac{1}{2} \int_{x_0}^{x_0+x_0} \frac{(1 - 3\sin^2\theta_1)(x - x_0)^2 dx}{r_0^3} + \int_{x_0}^{x_0+x_0} \frac{\cos\theta_1 x dx}{r_0^3} \right] \right. \\ \left. + \omega_s \left[ \frac{1}{6} \int_{x_0}^{x_0+x_0} \frac{(1 - 3\sin^2\theta_1)(x - x_0)^3 dx}{r_0^3} + \frac{x_0}{2} \int_{x_0}^{x_0+x_0} \frac{(1 - 3\sin^2\theta_1)(x - x_0)^2 dx}{r_0^3} \right. \right. \\ \left. \left. - x_0 \int_{x_0}^{x_0+x_0} \frac{\cos\theta_1(x - x_0)dx}{r_0^3} - \frac{1}{2} \int_{x_0}^{x_0+x_0} \frac{\cos\theta_1(x - x_0)^2 dx}{r_0^3} \right] + f(x_p, x_0, x_1, \omega) \right\}$$
(12)

The integrals in the above can all be computed. We then let  $x_p \rightarrow \infty$ , and include the odd terms containing  $x_p$  in function  $f$ . Then after lengthy but uncomplicated calculations, we finally get the very simple formula

where

$$v_x = (v_x)_{y=0} - \frac{\Gamma}{4\pi} (A\omega_y + B\dot{\omega}_y) \quad (13)$$

$$A = \left( x_0 + \frac{x'_1}{2} \right) \frac{\sqrt{x_1'^2 + x_1^2} + x'_1}{x_1'^2} - \frac{1}{2} \ln(\sqrt{x_1'^2 + x_1^2} - x'_1) \quad (14)$$

$$B = \frac{x_0 + x'_1}{2} \ln(\sqrt{x_1'^2 + x_1^2} - x'_1) - \frac{x'_1(x'_1 + 3x_0)(\sqrt{x_1'^2 + x_1^2} + x'_1)}{6x_1'^2} + \frac{1}{3} \sqrt{x_1'^2 + x_1^2} \quad (15)$$

where

$$x'_1 = x_1 - x_0, \quad s'_1 = s_1 - s_0 \quad (16)$$

Notice that in Equation (13) the odd terms containing  $f(x_p, x_0, x_1, \omega)$  are omitted since they will automatically cancel out when the effect of the free vortex with equal circulation but opposite directions starting from symmetric points from the rear edges of the left and right wings is computed.

#### IV. VORTEX LATTICE METHOD

Divide the wing into  $n$  bands along the half wing span and then divide chordwise into  $m$  equal parts using equal percentage line. The right half wing is then divided into  $N = n \times m$  small lattices with the left half wing symmetric to it. In each lattice is placed a non-stationary horseshoe vortex as described in paper [1]. The only difference is that the section of free vortices behind the rear edge should be changed to the snake-like vortices described in our paper, as shown in Figure 2. The oblique adjacent vortex is placed at the  $1/4$  chord point of the lattice with the control point taken at the center  $3/4$  chord point. The lattice span  $l_0$  is a constant. According to the computational formula of paper [1], it is most convenient to take  $b = l_0/2$  as the characteristic length.

Denote  $\mathcal{A}_i$  as a linear operator. It's effect on the dimensionless circulation  $\Gamma_i(t)$  of the  $i$ th non-stationary yaw oscillation oblique horseshoe vortex will yield the induced velocity in the  $y$  direction at its  $i$ th control point. According to boundary condition (3), the problem is to solve the equations

$$\sum_{i=1}^{2N} \mathcal{A}_i(\Gamma_i) = -\alpha_i - \text{sign}(z_i)\omega_i x_i \psi \quad i = 1, 2, \dots, 2N \quad (17)$$

where  $\alpha_i, x_i, z_i$  are respectively the local angle of attack and coordinates of the  $i$ th control point.

Expand the unknown circulation  $\Gamma_i$  as

$$\Gamma_i = (\Gamma_{\alpha})_i + (\Gamma_{\psi})_i \omega_i + (\Gamma_{\alpha})_i \omega_i, \quad (18)$$

where the subscripts  $\alpha$  and  $\psi$  correspond respectively to the problem of the angle of attack and to the problem of upper inversion angle. Apparently,  $(\Gamma_{\alpha})_i$  is the stationary dimensionless circulation for the motion with no yaw.

Expand  $\mathcal{A}_i$  as

$$\mathcal{A}_i = \mathcal{A}_i^0 + \mathcal{A}_i^1, \quad (19)$$

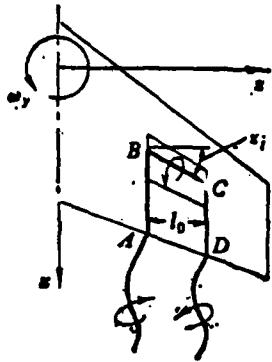


Figure 2

where  $\mathcal{K}_i$  represents the operator corresponding to the non-stationary oblique horseshoe vortex with no yaw and  $\mathcal{F}_i$  is the operator corresponding to the induced velocity increment caused by the change of the free vortices to snake-like ones along  $x$  direction behind the rear edge of the wing.

Obviously,

$$\begin{aligned}\mathcal{F}_i((\Gamma_{\infty}^+ + \Gamma_{\infty}^-), \omega_y) &= O(\omega_{\max}^3) \\ \mathcal{F}_i((\Gamma_{\infty}^+ - \Gamma_{\infty}^-), \dot{\omega}_y) &= O(\omega_{\max}^3)\end{aligned}$$

Substituting Equations (18), (19) into Equation (17) and then expanding, keeping only the first order term of  $\omega_{\max}$ , we then get

$$\begin{aligned}\sum_{i=1}^{2N} \{ &\mathcal{K}_i((\Gamma_{\infty})_i) + \mathcal{K}_i((\Gamma_{\infty}^+), \omega_y) + \mathcal{K}_i((\Gamma_{\infty}^-), \omega_y) \\ &+ \mathcal{K}_i((\Gamma_{\infty}^+), \dot{\omega}_y) + \mathcal{K}_i((\Gamma_{\infty}^-), \dot{\omega}_y) + \mathcal{F}_i((\Gamma_{\infty})_i)\} \\ &= -\alpha_i - \text{sign}(z_i) \omega_x \tau_i \psi \quad i = 1, 2, \dots, 2N\end{aligned}\quad (20)$$

According to Section III,

$$\mathcal{F}_i((\Gamma_{\infty})_i) = -\frac{1}{4\pi} ([A]_{ii} \omega_y + [B]_{ii} \dot{\omega}_y) (\Gamma_{\infty})_i \quad (21)$$

When the matrix coefficients  $[A]_{ii}$  and  $[B]_{ii}$  are computed with (14) to (16), we should notice the fact that there are two snake-like free vortices with opposite directions in the oblique horseshoe vortex model.

$$\mathcal{K}_i((\Gamma_{\infty})_i) = \frac{1}{4\pi} (\omega_y)_{ii} (\Gamma_{\infty})_i \quad (22)$$

where  $(\omega_y)_{ii}$  may be calculated according to Equations (3.6) (3.7) in paper [1].

Also according to paper [1], when  $\omega \rightarrow 0$ , we have

$$\mathcal{K}_i\left(\left(\frac{(\Gamma_{\infty})_i}{(\Gamma_{\infty})_i}\right) \omega_y\right) = \frac{1}{4\pi} \left[ (\omega_y)_{ii} \left(\frac{(\Gamma_{\infty})_i}{(\Gamma_{\infty})_i}\right) \omega_y + \left[\left(\frac{\partial \omega_y^{(2)}}{\partial \omega}\right)_{\omega=0}\right]_{ii} \left(\frac{(\Gamma_{\infty})_i}{(\Gamma_{\infty})_i}\right) \dot{\omega}_y \right] \quad (23)$$

$$\Re_i \left( \left( \frac{(\Gamma_y)}{(\Gamma_x)} \right)_{ii} \omega_y \right) = \frac{1}{4\pi} (\omega_y)_{ii} \left( \frac{(\Gamma_y)}{(\Gamma_x)} \right) \dot{\omega}_y \quad (24)$$

In the above equations,  $(\omega_y)_{ii}$  and  $\left( \frac{\partial \omega_y^{(i)}}{\partial \omega} \right)_{ii}$  are calculated with Equations (3.7) and (3.25) in paper [1].

Substituting Equations (21) to (24) in Equation (20), separating the problems of the angle of attack and that of the upper inversion angle and then separately equating the constant terms, the sine terms (terms containing  $\dot{\omega}_y$ ) and cosine terms containing  $\omega_y$  on the two sides, we get the following equations:

For problem of the angle of attack:

$$\begin{aligned} \sum_{i=1}^{2N} (\omega_y)_{ii} (\Gamma_{x,i})_i &= -4\pi a_i \\ \sum_{i=1}^{2N} (\omega_y)_{ii} (\Gamma_y)_i &= \sum_{i=1}^{2N} [A]_{ii} (\Gamma_{x,i})_i \\ \sum_{i=1}^{2N} (\omega_y)_{ii} (\Gamma_y)_i &= \sum_{i=1}^{2N} [B]_{ii} (\Gamma_{x,i})_i - \sum_{i=1}^{2N} \left[ \left( \frac{\partial \omega_y^{(i)}}{\partial \omega} \right)_{ii} \right] (\Gamma_y)_i, \quad i = 1, 2, \dots, 2N \end{aligned} \quad (25)$$

The three sets of linear algebraic equations above may be solved one by one. The first set of equations correspond to the problem of no yaw symmetry. Thus, we only need to solve for the N unknowns on the right wing. It may be proved that the second and third set of equations correspond to problems of anti-symmetry and also need only be solved for the N unknowns on the right wing.

Problem of upper inversion angle:

$$\begin{aligned} \sum_{i=1}^{2N} (\omega_y)_{ii} (\Gamma_y)_i &= -4\pi \text{sign}(z_i) z_i \psi \\ \sum_{i=1}^{2N} (\omega_y)_{ii} (\Gamma_y)_i &= - \sum_{i=1}^{2N} \left[ \left( \frac{\partial \omega_y^{(i)}}{\partial \omega} \right)_{ii} \right] (\Gamma_y)_i, \quad i = 1, 2, \dots, 2N \end{aligned} \quad (26)$$

<sup>1)</sup> The typographic errors in 2 arithmetic signs of Equation (3.25) in paper [1] have been corrected.

These two sets of equations are both anti-symmetric and require only half wing solution.

## V. AERODYNAMIC LOAD COMPUTATION

It is not difficult to prove from the pressure coefficient Equation (2) that the Joukowski theorem still holds for non-stationary oblique horseshoe vortices. Consider the left produced by the jth vortex model of the right wing as in Figure 2.

The left of BC vortex is:

$$(Y_{BC})_j = \rho_a(V_\infty + V_a\omega, z_j)(V_a b \Gamma_j) l_0 \\ + \rho_a(V_a\omega, z_j)(V_a b \Gamma_j) l_0 \tan \chi_j$$

where  $x_j, z_j, \chi_j$  are the coordinates and backswept angle of the center point of the vortex. Substituting in the above Equation (18) and keeping only the first order term of  $\omega_{y_{max}}$ , we get

$$(Y_{BC})_j = \rho_a V_a [V_a b(\Gamma_{a,0})_j] l_0 + \rho_a V_a \{V_a b(\Gamma_{a,y} + \Gamma_{a,z})_j \omega_y \\ + V_a b(\Gamma_{a,y} + \Gamma_{a,z})_j \dot{\omega}_y + \rho_a (V_a \omega, z_j) [V_a b(\Gamma_{a,0})_j] l_0 \\ + \rho_a (V_a \omega, z_j) [V_a b(\Gamma_{a,0})_j] l_0 \tan \chi_j + O(\omega_{y_{max}}^2)\} \quad (27)$$

where the first term does not contribute to the rolling moment.

The left of CD vortex is:

$$(Y_{CD})_j = \rho_a \int_{z_j}^{x_D} (V_a \omega, x) (V_a b \Gamma_j) b dx \\ = \rho_a V_a^2 b^2 \omega, (\Gamma_{a,0})_j \left( \frac{x_D^2 - z_j^2}{2} \right) + O(\omega_{y_{max}}^2)$$

Similarly, the left of BA vortex is

$$(Y_{BA})_j = -\rho_a V_a^2 b^2 \omega, (\Gamma_{a,0})_j \left( \frac{x_D^2 - z_j^2}{2} \right) + O(\omega_{y_{max}}^2)$$

It may be seen that, different from the case of no yaw, with yaw, the vortices CD and BA are also left-producing adjacent vortices and snake-like free vortices which do not produce lift are only found behind the rear edge of the wing. The rolling

moment around the x axis may be found by summing the contributions of all the vortices.

## VI. COMPUTATIONAL RESULTS

With the wing surface and span as characteristics, define the rolling moment coefficient  $m_x$ . Again with the half wing span and  $V_\infty$  as characteristics, define the dimensionless time  $t$  and dimensionless angular velocity  $\omega_y$ . According to the customary method to express aerodynamic derivatives:

$$m_x^* = \frac{\partial m_x}{\partial \omega}, \quad m_x^{*'} = \frac{\partial m_x}{\partial \omega'}, \quad m_x'' = \frac{\partial m_x}{\partial \beta}, \quad m_x''' = \frac{\partial m_x}{\partial \dot{\beta}}$$

where  $\beta$  is the slip angle,  $\dot{\beta} = d\beta/dt$ .

For many trapezoidal backswept wings, we calculate for the two center of gravity positions  $\bar{s}_r = 0.25$  and 0.5 (ratio of the distance from the front edge of the average aerodynamic chord  $b_A$  relative to the rotational center to  $b_A$ ).  $m_x^*$  and  $m_x^{*'}$  are found by using the transformation equations of the aerodynamic derivatives when the center of gravity is shifted. Part of the numerical result and empirical estimation of the aerodynamic derivatives are compared to the experimental result. Except when particularly noticed, we use  $n \times m = 8 \times 4$  for calculation in this paper. Let  $\lambda$  be the wing span chord ratio,  $n$  be the root tip ratio and  $\chi$  be the quarter chord line backswept angle. Figures 3 to 5 are the cases of plane wing with angle of attack ( $\psi = 0$ ), with  $c_y$  as the lift coefficient. Figures 6 and 7 are cases with upper inversion angle ( $\alpha = 0$ ). Except for Figure 7 where  $\psi$  is measured in degrees, radians are used in all other cases. Figure 3 has been transformed to relative stationary axes system.

Relative body axes system is used for the rest.

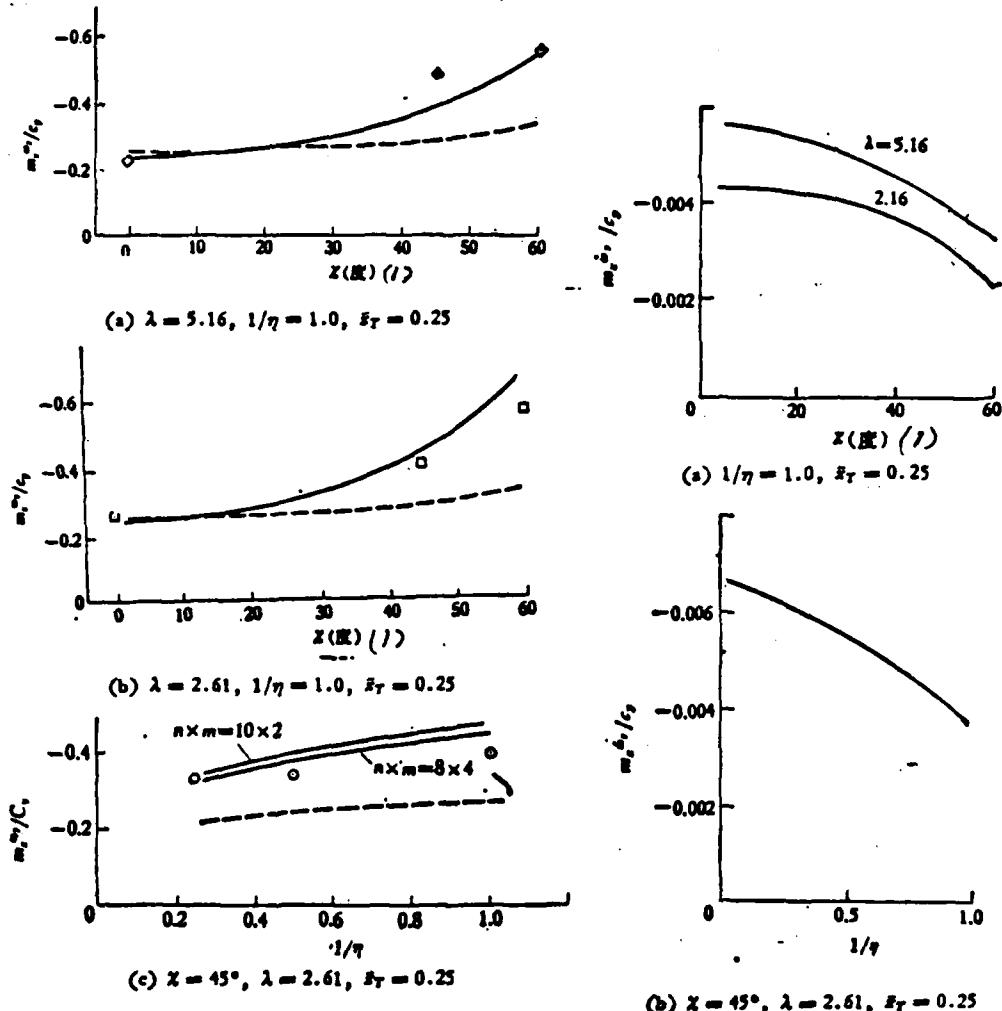


Figure 3. Computational result for  $m_0^y/e_y$  (relatively stationary axes)

—this paper --- estimation paper [2]

- ◆ paper [3]
- paper [3] experiment
- paper [4]

Key: (1) degrees

Figure 4. Computational result for  $m_0^y/e_y$



Computations proved that for problems of angle of attack, the effect on the result of whether or not to consider the change of wake vortices into snake-like is small, about 10%. However, consideration of this effect will make the computational result come out closer to experiment. The time difference derivative in problems of angle of attack is entirely caused by the change of wake vortices to snake-like. Its value is very small.

For triangular wing with very small span-chord ratio, the effect of the wake vortices on the wing may be neglected. Then  $\phi$  is equivalent to the theoretical perturbation velocity potential of a long thin body in problems of stationary angle of attack with no yaw. Here Equation (2) becomes the theoretical pressure coefficient equation of a thin long body in paper [6]. Hence, our method approaches the theory for a long thin body for this case.

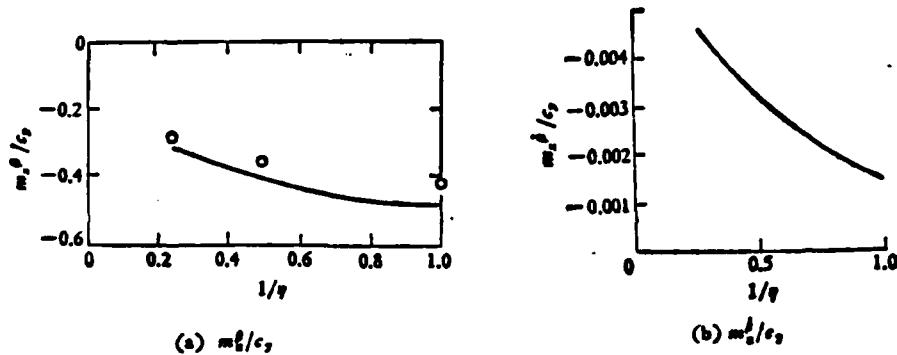


Figure 5. Yaw derivatives for a plane wing  
 $\lambda = 2.61, \chi = 45^\circ$  —this paper; o experiment, paper [4]

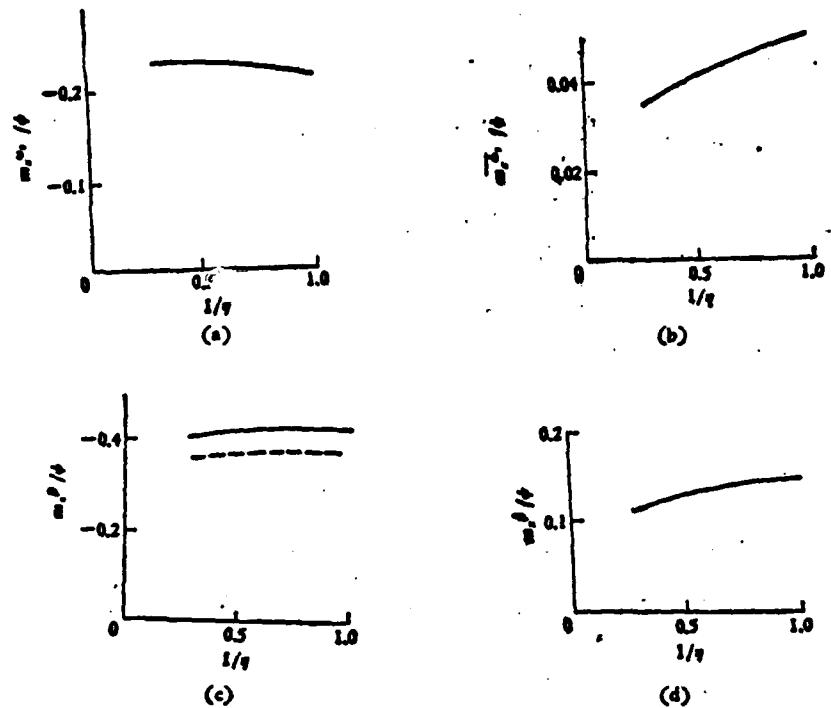


Figure 6. Computational result for the problem of upper inversion angle

$\chi = 45^\circ$ ,  $\lambda = 2.61$ ,  $\bar{x}_T = 0.25$

—this paper; —Weissinger extended lift surface theory

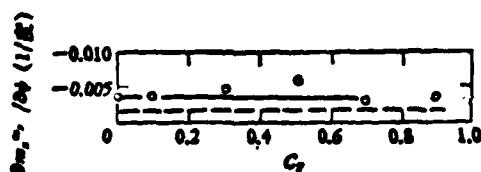


Figure 7. Comparison of computational and experimental results for the problem of upper inversion angle

$\chi = 45^\circ$ ,  $\lambda = 2.61$ ,  $1/n = 1.0$ ,  $\bar{x}_T = 0.25$

—this paper; ---estimate, paper [2]; o experiment, paper [5]

## VII. CONCLUSIONS

In this paper we have only established the lift surface theory for wings in yaw oscillation at low speed and provided the numerical method of solution for the case of low frequency oscillation. By separating the problem into several symmetric and antisymmetric problems, we have realized large savings in computer internal storage and in computer time.

The effect of a side edge of a wing changing into a rear edge during yaw and side slip has not been considered in our method, hence when the effect of side edge becomes very important (e.g., for trapezoidal wing with very small span-chord ratio), this method is not applicable.

## REFERENCES

- [1] Belotserkovskiy, S.M., Skripach, B.K. Derivatives of an Aircraft and Wing at Subsonic Velocities. "Nauka" Press, Moscow, 1975.
- [2] Toll, Thomas. A., Queijo, M. J., NACA, TN 1581 (1948).
- [3] Goodman, Alex and Brewer, Jack. D., NACA, TN 1669 (1948).
- [4] Letke, William and Cowan, John. W., NACA, TN 1671 (1948).
- [5] Queijo, M. J. and Jaquet, Syree. M., NACA, TN 1668 (1948).
- [6] Ribaer, Herbert. S., NACA, TN 1423 (1947).

## Abstract

This paper sets up the mathematical problem of flow past a yawing oscillating wing at low speeds by using a set of axes fixed in the wing. The snake-like vortices behind the wing are investigated in detail. In the case of low-frequency oscillation, the problem is solved numerically by the nonsteady vortex-lattice method. This method can be applied to calculate rolling-moment derivatives of wing due to yaw and sideslip. Some of the results obtained are compared with experimental data.

# SIMPLIFIED NAVIER-STOKES EQUATIONS AND THEIR MATHEMATICAL PROPERTIES\*

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## ABSTRACT

In this paper, the two-dimensional simplified Navier-Stokes equations are analyzed qualitatively. The authors conclude that the above equations are ill-posed when the tangential flow velocity component  $u < c(c)$  is the local sound velocity. Hence, instability arises in numerical solutions. In addition, the proper specification of initial and boundary value conditions of supersonic viscous flow around a sphere-cone body is also discussed.

Ever since 1904 when L. Prandtl proposed the boundary layer theory, it has greatly affected the research in viscous flow but following the development of high speed machines, the boundary layer theory has become insufficient to treat many important practical problems. Direct numerical solution of the complete Navier-Stokes equations (abbreviated as N-S equations below) has reaped fruitful result but the volume of computation and the amount of computational time are still sizeable. There is a way that has attracted people's attention, namely to improve on the boundary equations or to simplify the complete N-S equations in order to reach the goal of solving effectively practical problems. Much work has been done in this direction since the 60's which may be classified into three categories: 1. establishment of second order boundary layer theory;<sup>L1</sup> 2. thin shock wave layer approximation,<sup>L2, L3</sup> and 3. simplification of the N-S equations.<sup>L4-L7</sup> We would like to point out that when  $Re \rightarrow \infty$  the

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thin shock wave layer equations cannot transform smoothly into the Euler equations while the simplified N-S equations will transform smoothly to the non-viscous equations for  $Re_\infty \rightarrow \infty$ ; secondly, the simplified N-S equations may be solved with a unified computational scheme without the need to use the complicated method for solving the boundary layer equations.

Much work has been done in solving a series of practical problems with the simplified N-S equations and the common problem encountered is the instability in the numerical solutions that has appeared when the initial boundary value problem is solved.<sup>[23-27]</sup> Many people have proposed various methods of avoiding such difficulties but few have investigated mathematically the cause of the instability, the properties of the equations and the proper specifications of the boundary value conditions. Our paper qualitatively analyzes these questions from the mathematical and mechanical point of view, based on the practical computational experiences of the authors, with the purpose of discussing it with scholars both at home and abroad.

## 1. SIMPLIFIED N-S EQUATIONS

The form of simplified N-S equations varies depending on how the RHS is chosen. The practical difference is very small when the Reynolds number  $Re_\infty$  is large. In the following, we shall write down the simplified N-S equations [27] derived by Gao Zhi in 1967 when he analyzed the problem of viscosity--non-viscous flow interference (see Figure 1).

$$\frac{\partial}{\partial x} (r \rho u) + \frac{\partial}{\partial y} (r H \rho v) = 0 \quad (1.1)$$

$$\begin{aligned} \rho \left( \frac{u}{H} \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + k \frac{uv}{H} \right) + \frac{1}{H} \frac{\partial p}{\partial x} \\ - \frac{1}{Re_\infty} \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) + v \frac{\mu}{Re_\infty} \frac{\partial u}{\partial y} - k \frac{\mu}{Re_\infty} \frac{\partial}{\partial y} \left( \frac{\mu}{H} \right) \end{aligned} \quad (1.2)$$

$$\rho \left( \frac{u}{H} \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} - k \frac{v^2}{H} \right) + \frac{\partial p}{\partial y} = 0 \quad (1.3)$$

$$c_s \rho \left( \frac{u}{H} \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) - \left( \frac{u}{H} \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} \right) = \delta \frac{\partial}{\partial y} \left( \mu \frac{\partial T}{\partial y} \right) + \delta \pi \mu \frac{\partial T}{\partial y} + \phi \quad (1.4)$$

$$p = \rho T \quad (1.5)$$

where  $H = 1 + ky$ ,  $x = \frac{\cos \theta}{r} + \frac{k}{H}$ ,  $\phi = \frac{\mu}{Re_\infty} \left[ \left( \frac{\partial u}{\partial y} \right)^2 - 2k \frac{u}{H} \frac{\partial u}{\partial y} + \frac{4}{3} \left( \frac{\partial v}{\partial y} \right)^2 \right]$ ,

$c_s$ ,  $\rho$ ,  $T$ ,  $\mu$ ,  $k$  represent respectively the pressure, density, temperature, viscosity coefficient and thermal conductivity. Converting to dimensionless quantities, for  $x, y, r$  we use  $a$ ; for  $u, v$  we use  $V_\infty$  and for  $p, \rho, T, \mu, k$  we use respectively  $\rho_\infty V_\infty^2, \rho_\infty, V_\infty^2/R, \mu_\infty$  and  $\lambda_\infty$ . The Equations (1.1)-(1.5) include the non-viscous equation which differs fundamentally from the boundary layer equations in its mathematical type. This will be discussed below.

## 2. TYPE OF SIMPLIFIED N-S EQUATIONS

As is well known, the classical boundary layer equations belong to the parabolic type and are well-posed to be solved as initial value problems. There is no unified way of specifying the simplified equations. Some people propose parabolic type equations [13-15] and some parabolic-hyperbolic equations [16,17]. Paper [19] is of the opinion that in the vicinity of the object surface (assuming the wall conditions are  $u=0, v=0, T=T_w$ ) we have the property of reverse thermal conduction. The fundamental type of the equations determines the way information is propagated in the flow field as well as the specification of the boundary conditions and the numerical solution, therefore, it is a very important question.

The simplified N-S equations consist of several first order and several second order quasilinear partial differential equations. It is difficult to ascertain its type in the strict classical sense. Since the equations contain both convection effect and diffusion effect, we can analyze the behavior of the solution using one of the effects as the principle one.

In view of the quasilinearity of the equations, we limit our discussion to a small region about a point. When  $\mu \neq 0$ , we introduce the following auxiliary variable

$$\tau = \frac{1}{Re} \mu \frac{\partial u}{\partial y}, q = \delta \mu \frac{\partial T}{\partial y} \quad (2.1)$$

Then we can convert (1.1)-(1.5) to first order equations

$$A \frac{\partial X}{\partial x} + B \frac{\partial X}{\partial y} = F \quad (2.2)$$

where A, B are 6x6 matrices, X, F are 6 dimension column vectors. The characteristic equation of (2.2) is

$$\det(\sigma_1 a_{ij} + \sigma_2 b_{ij}) = 0 \quad i, j = 1, 2, \dots, 6 \quad (2.3)$$

with characteristic root

$$\sigma_1 = 0, \lambda_1 = H \left( \frac{v}{u} + \frac{1}{u} \sqrt{p/\rho} \right), \lambda_2 = H \left( \frac{v}{u} - \frac{1}{u} \sqrt{p/\rho} \right) \quad (2.4)$$

The result agrees with the conclusions of [16,17].

From (2.4), it can be seen that the differential equations have four multiple eigenvalues, reflecting the parabolic property with the signal propagating along  $x = \text{constant}$  with infinite speed; there are two other non-zero real eigenvalues, representing the hyperbolic property with the signal propagating along the characteristic directions with finite speed. Hence, when  $\mu \neq 0$  or when diffusion dominates, we may regard the simplified N-S equations as a set of equations with both parabolic and hyperbolic properties. The same conclusion may be reached with the method of [25].

On the other hand, when  $\mu = 0$ , i.e., when (1.1)-(1.5) are reduced to first order equations, the characteristic roots are

$$\lambda_1 = \lambda_2 = Hv/u, \lambda_{3,4} = H \frac{uv \pm c^2 \sqrt{M^2 - 1}}{u^2 - c^2} \quad (2.5)$$

where  $M$  is the Mach number. When  $M > 1$ , all the characteristic roots are real and the equations are hyperbolic; when  $M < 1$ ,  $\lambda_3$  and  $\lambda_4$  are complex roots and the equations are elliptic. Hence, when the diffusion effect is very small, ( $\mu \rightarrow 0$ ), the property of the equations will undergo fundamental changes. The property of the equations in the intermediate stages is complex but we think the more important question is whether the specification of initial-boundary values for the Equations (1.1)-(1.5) is well posed.

### 3. THE WELL-POSEDNESS OF INITIAL VALUE PROBLEMS

In [16,17] it has been pointed out the hyperbolic-parabolic property of the simplified equations but from this they reached the conclusion: it is well-posed to integrate downstream as an initial boundary value problem from the stagnation point in the flow field of a supersonic flow around a blunt object. We take a different view. We discover after some analysis that when  $u < c$ , the Cauchy problem is ill-posed. For simplicity, let us consider the major components of the original equations when the coefficients are frozen:

$$A \frac{\partial X}{\partial x} + B \frac{\partial X}{\partial y} - C \frac{\partial^2 X}{\partial y^2} \quad (3.1)$$

where  $X^r = (u, v, p, T)$

$$A = \begin{pmatrix} \frac{1}{H} & 0 & \frac{\mu}{H\rho} & -\frac{\mu}{HT} \\ 0 & \frac{\rho\mu}{H} & 0 & 0 \\ \frac{\rho\mu}{H} & 0 & \frac{1}{H} & 0 \\ 0 & 0 & -\frac{\mu}{H} & c_p \frac{\rho\mu}{H} \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 1 & \frac{\rho}{\rho} & -\frac{\rho}{T} \\ 0 & \rho\nu & 1 & 0 \\ \rho\nu & 0 & 0 & 0 \\ -\frac{\mu}{Re} \mu_r & 0 & -\nu & c_p \rho\nu \end{pmatrix};$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\mu}{Re} & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta\mu \end{pmatrix}.$$

When  $u = 0$  and  $u = c$ , the matrix A is degenerate; therefore, it is not possible to explicitly solve for  $\frac{\partial X}{\partial x}$  from Equation (3.1). In order to discuss whether the Cauchy problem is well-posed when  $x > 0$ , we notice the following theorem [20]

The necessary condition that the equations with constant coefficients

$$\frac{\partial X}{\partial t} = A \frac{\partial X}{\partial x} + B \frac{\partial^2 X}{\partial x^2} \quad (3.2)$$

is non-degenerate is that the real part of the characteristic values of matrix B be positive. If these characteristic values are different from each other, then the condition is also sufficient.

We use the same method as in [20] to derive that the necessary condition that (3.1) is non-degenerate is that the roots of its characteristic equation

$$\det(\lambda A - C) = 0 \quad (3.3)$$

have non-negative real parts. The roots of (3.3) are

$$\begin{aligned} \lambda_1 &= \lambda_2 = 0 \\ \lambda_{3,4} &= \frac{\mu}{2Re_m} \cdot \frac{1}{Pr} \cdot \frac{H}{\rho\mu(m^2 - 1)} \{ [(Pr + \gamma)m^2 - 1] \\ &\quad \pm \{ [(Pr + \gamma)m^2 - 1]^2 - 4Pr\gamma m^2(m^2 - 1) \}^{1/2} \} \end{aligned}$$

where  $m^2 = (u/c)^2$ . It is not difficult to see that when  $u \geq c$ ,  $\lambda_1, \lambda_2 > 0$ ; and when  $u < c$ ,  $\lambda_1 < 0, \lambda_2 > 0$ . Therefore, when  $u < c$ , (3.1) do not satisfy the necessary condition for being non-degenerate. The initial value problem is then ill-posed.

The fact that computational instability appears in the region  $u < c$  has been mentioned in many works [13-15,19]. We think that the instability in the difference solution is caused by the ill-posedness of the original partial differential equations when  $u < c$ . No instability appears in the region  $u \geq c$  for the same implicit difference scheme. From a mathematical point of view, the key is that the coefficient matrix A of the derivative  $\partial X/\partial x$  in Equation (3.1) becomes degenerate in the solution region. Hence, to overcome the instability in the numerical solution, we must change the structure of matrix A. For example, we may treat  $\partial p/\partial x$  [13] explicitly in the x momentum equation, treat  $\partial u/\partial x$  [19] explicitly in the continuity equation, choose  $\partial v/\partial x$  [5] in the y momentum equation and use special method [26] to prevent the development of perturbation, etc. In fact, if we regard  $\partial p/\partial x$  in x momentum equation or  $\partial u/\partial x$  in the continuity equation as relatively fixed, then the characteristic equation  $\det(\lambda A - C) = 0$  of Equation (3.1) has roots

$$\lambda_1 = \lambda_2 = 0; \quad \lambda_3 = \frac{H}{\rho u} \left( \frac{\mu}{Re} \right) > 0; \quad \lambda_4 = \frac{\gamma}{Pr} \lambda_3 > 0. \quad (3.4)$$

with non-negative real parts, hence satisfying the necessary condition of non-degenerate equations.

#### 4. SPECIFICATION OF INITIAL-BOUNDARY CONDITIONS

As mentioned before, the simplified N-S equations have an ill-posed initial value problem in the region  $u < c$ . But then, how should the boundary conditions be specified so as to guarantee that the problem is well-posed? In the following we shall only qualitatively analyze the supersonic viscous flow around a blunt body.

In non-viscous flow, the flow diagram is as shown in Figure 2. If the initial value problem is specified from the stagnation point, as is done in integral relation method [21], then we have two problems. First, the stagnation point solution cannot be independently established. It is related to the normal condition through the singular line  $u = c$  in the flow field; second, it is unstable to integrate downstream from the stagnation point. This instability arises because the problem is ill-posed.

For viscous flow, if the surface conditions are  $\alpha = 0, \epsilon = 0, T = T_w$  (no slip stream and no injection) then the local sound velocity line  $u = c$  in the flow field will not intersect the object surface but will extend closely along the body all the way to the point E (Figure 3) near the tail region at the bottom. In other words, the subsonic region at the top is connected to the subsonic return flow region at the bottom by a very thin subsonic band near the object surface. Will the signals before and after passing through this band affect each other? This is a problem of much argument among scholars in foreign countries. Most people think that this effect exists [13-15,19,22], but some argue from the fact that according to the analysis from the point of view of mechanics the signals after will not affect those before, hence the specification of initial boundary problems for the simplified N-S equations is well-<sup>[23,24]</sup> posed. According to our earlier analysis, the former view is more reasonable. As a matter of fact, assuming that the simplified N-S equations are solved by using the integral relation method, since the curve  $u = c$  does not fall on the object surface and falls instead in the bottom region, we may imagine that the number  $N$  of the strips in the integral relation method approach infinity so that the infinite many unknowns to be determined on the stagnation line are associated with the infinite number of regularity conditions on  $u = c$ . When  $N \rightarrow \infty$ , the quantity on the stagnation line  $x = 0$  is associated with the regularity condition on the whole line  $u = c$  which means that the "relation" between the signals before and after does

exist. In principle, the solution must be sought as a boundary value problem in the region A B C D E F A and it is unreasonable to find the solution along the x direction by artificially fixing the flow field at the beginning.

According to the above analysis, we think the boundary conditions for solution in the region A B C D E F A may be given as follows: Specify symmetric condition  $x = 0$ , no condition on the EF line, and any additional conditions not satisfied at  $x=0$  will be supplemented by the regularity conditions of the implicit solutions on curve  $u = c$ . The y direction boundary conditions may be determined by the characteristics of the original partial differential equations. From Section 2, we know that the equations have 4  $x = \text{const}$  characteristics, representing the parabolic property. Signals propagate along  $x = \text{const}$  with infinite velocity. Two boundary value conditions must be given on  $y = 0$  and  $y = y_s$ . Also, since the characteristic values  $\lambda_1(0) > 0, \lambda_2(0) < 0$ , therefore, one condition needs be given at  $y = 0$ .  $\lambda_1(y_s) > 0, \lambda_2(y_s) < 0$ , therefore, again one condition needs be given at  $y = y_s$ . If we let the detached shock wave of the flow around the blunt body be the unknown, then one additional condition must be specified for a total of seven conditions. When we take three conditions on the object surface and four conditions on the shock wave, the equations happen to be closed but if we keep the second order derivative of v with respect to y in the normal direction momentum equation, then we lack one boundary condition which is generally supplemented by the continuity equation on the shock wave [14] or the normal momentum equation on the wall surface.

In practical computation, for a narrow, long blunt body, the flow field at the bottom and at the top interact very weakly (the perturbation gradually disappears due to viscous dissipation). Therefore, it is possible to divide the solution region into four regions I, II, III, IV. I is the blunt nose region, mostly subsonic;

II is the body region flow field, mostly supersonic with the vicinity of the wall a thin subsonic region. III is the near-tail flow region with a very complicated flow structure; IV is the far tail flow region, i.e., all supersonic region. The criteria used to separate the four regions above is that the interaction between the solutions of the two neighboring regions should be negligible. We use the linear method to find the solution in region I. Experience indicates that if the line ab includes the major subsonic part, then the effect of region II on region I may be neglected. Region II then can be solved as an initial boundary problem. But there exists the numerical instability. A stable numerical solution can be obtained simply by changing the property of the coefficient matrix A as pointed out in Section 3, or by suitably controlling the step length. Region III should be solved as a boundary value problem. Many people still use equations of the type (1.1)-(1.5) but it is more reasonable to keep the second order derivative term along x direction. As for region IV, it is an all-supersonic region. It is reasonable to solve it as an initial boundary value problem by using Equations (1.1)-(1.5). Our results for numerical solution of regions I and II for a sphere-cone composite object indicates that the above method of region separation is workable, much more effective than the method of R. T. Davis [5] or time relation method. At present, simplified N-S equations have been applied to channel flow [8], near tail flow [9], far tail flow [10], compressed corner [11], shock wave attached surface layer perturbation problem [12] and viscous flow around three-dimensional sharp and blunt nosed bodies, and the effect is good.

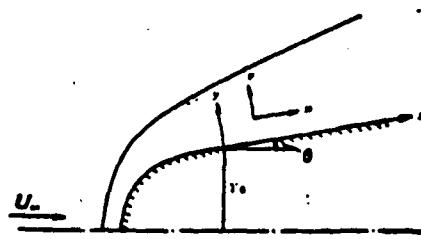


Figure 1

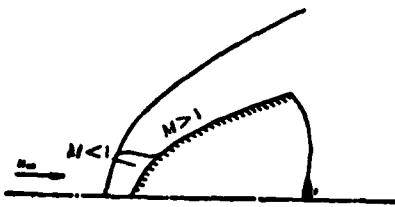


Figure 2

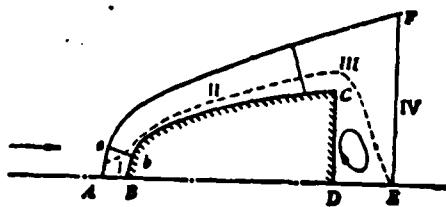


Figure 3

## REFERENCES

- [1] Van Dyke, N., *Hypersonic Flow Research*, Academic Press, N. Y. (1963).
- [2] Cheng, H. K., IAS paper, 63-92 (1963).
- [3] Магомедов, К. М., Изв. АН СССР. МЖГ, 2(1970), 44-56.
- [4] Головачев, Ю. П. и др., ЖВММФ, 13, 4(1978), 1021-1028.
- [5] Davis, B. T., AIAA J., 8, 5 (1970), 843-851.
- [6] Толстых, А. И., ЖВММФ, 6, 1(1966), 113-120.
- [7] Srivastava, B. N. et al., AIAA J., 14, 2(1976), 243-245.
- [8] Ветукин, В. Н., Изв. АН СССР. МЖГ, 4(1977), 29-35.
- [9] Ohrenberger, J. et al., AIAA paper, (1973), 72-118.
- [10] Кокошская, Н. С. и др., *Аэродинамика и Газовая Динамика*, М., «Наука» (1976).
- [11] Miller, D. G. et al., AIAA Paper, 75-1(1975).
- [12] Lee, R. I. et al., AIAA J., 15, 3 (1977), 189-188.
- [13] Rubin, S. G. et al., *Computer and Fluids*, 1, 1 (1973), 37-57.
- [14] Hellinwell, W. S. et al., *Computer and Fluids*, 1 (1975), 83-101.
- [15] Patankar, S. V. and Spalding, D. B., *Int. J. Heat mass Transfer*, 15, 10 (1972), 1787-1805.

- [16] Davis, R. T. and Flügge-Lotz, I. *J. Fluid Mech.*, 20, part (1954), 539—523.
- [17] Кайзер, Й. Е. и Flügge-Lotz, I., AD—669578 (1968).
- [18] Петровский, И. Г., Лекции об уравнениях с частными производными, М., Физматгиз (1961).
- [19] Воронкин, В. Г., Изв. АН СССР, МЖГ, 6(1974), 99—105.
- [20] Галифанд, И. М., УМН., 14, 2(1959), 87—158.
- [21] Белоцерковский, О. М., Численные методы решения задач механики сплошных сред, Вып. АН СССР, (1969).
- [22] Garvine, R. W. *The Phys. Fluids*, 11, 7 (1968), 1413—1423.
- [23] Müller, G., AD—746891 (1972).
- [24] Müller, G., *AIAA J.*, 11, 7(1973), 933—943.
- [25] Абринич, В. Н., ДАН БССР, 20, 11(1976), 968—970.
- [26] Конев, В. М., Черныш, С. Г., Яненко, Н. Н., ДАН СССР, 245, 6(1979), 1522—1524.
  
- [27] Gao Zhi; Simultaneous Solution of non-Viscous External Flow and Viscous Boundary Layer, Work Report (1967), Mechanical Research Institute, Academia Sinica

#### Abstract

This paper presents some qualitative analysis for simplified Navier-Stokes equations. Authors conclude that the system of equations is ill-posed when tangential component of velocity  $\mathbf{u}$  is smaller than  $c$  ( $c$  is local sound velocity) in the flow field. Therefore, instabilities have been encountered in numerical solutions. In addition, we also discuss the problem of proper specification of the initial-boundary value for viscous hypersonic flow around a sphere-cone body.

EQUATIONS OF FLIGHT MECHANICS CONSIDERING  
ATMOSPHERIC TURBULENCE

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To study the effect of wind on the flight performance, the general form of the flight mechanics equation with wind must be given [1] - [6]. Starting from the difference between the relative motion and the absolute motion, this paper derives the general form of the flight mechanics equations with wind. Let  $\mathbf{G}, \mathbf{V}_a, \mathbf{V}_r, \mathbf{W}, \mathbf{A}, \mathbf{T}, \mathbf{M}, \omega_a, \omega_r, \Omega$  be the vector forms of the weight, air speed, ground speed, wind speed, aerodynamic force, thrust, aerodynamic moment, angular velocity relative to the air, angular velocity relative to the ground of the flying vehicle and the angular velocity of the air respectively. The body coordinate system is  $Oxyz$  and the inertial coordinate system is  $O_gx_gy_gz_g$ . The components of the above vectors along the coordinates will be indicated by the corresponding subscripts  $\phi, \theta, \gamma$  are the angles between the body coordinates and the inertial coordinates. The cyclic order is  $\phi \rightarrow \theta \rightarrow \gamma; \alpha$ .  $\alpha$  is the angle of attack,  $\beta$  is the sideslip angle,  $\phi$  is the drift angle,  $\theta$  is the flight path angle,  $\mu$  is the flow deviation angle,  $a$  is the sound speed,  $J$  is the rotary moment of inertia (which components will be indicated by the corresponding subscripts),  $m$  is the mass of the flying vehicle,  $M$  is the Mach number,  $H$  is the pressure head,  $Q$  is the flow rate of the fuel. After certain simplification, the vectorial form of the flight mechanics equations with wind are

$$\left. \begin{aligned} m d\mathbf{V}_r / dt &= \mathbf{G} + \mathbf{A}(\mathbf{V}_a, \alpha, \beta, H, M, \dots) + \mathbf{T}(Q, M, H, \dots) \\ J d\omega_r / dt &= \mathbf{M}(\mathbf{V}_a, \alpha, \beta, \omega_a, M, \dots) \\ \mathbf{V}_r &= \mathbf{V}_a - \mathbf{W}, \quad \omega_r = \omega_a - \Omega \end{aligned} \right\} \quad (1)$$

The initial conditions are  $V_s|_{t=0} = V_s(0), \omega_s|_{t=0} = \omega_s(0)$ .

The dynamic equation contains both the absolute parameters of motion  $V_g$  and  $\omega_g$  as well as the relative parameters of motion  $V_a$  and  $\omega_a$ . Since

$$V_s = W + V_a, \quad \omega_s = Q + \omega_a,$$

Hence

$$\begin{aligned} dV_s/dt &= (dV_a/dt) + (dW/dt) + 2Q \times V_a; \\ d\omega_s/dt &= (d\omega_a/dt) + (dQ/dt). \end{aligned}$$

After multiplication with  $m$  and  $J$  respectively and substituting into (1), the following system of equations are obtained:

$$\left. \begin{aligned} m \frac{dV_a}{dt} &= G + A(V_a, \alpha, \beta, H, M, \dots) + T(Q, H, M, \dots) \\ &\quad - m \frac{dW}{dt} - m 2Q \times V_a \\ J \frac{d\omega_a}{dt} &= M(V_a, \alpha, \beta, \omega_a, M, H, \dots) - J \frac{dQ}{dt} \\ V_s &= W + V_a, \quad \omega_s = Q + \omega_a. \end{aligned} \right\} \quad (2)$$

The initial conditions are  $V_s|_{t=0} = V_s(0), \omega_s|_{t=0} = \omega_s(0)$ .

In order to solve the systems of equations (1) and (2), they must be transformed into their corresponding scalar forms first.

To simplify the transformation of the vectors, the moving coordinate system is taken to be the same as the body axis coordinate system Oxyz. The relation between  $V_a$  and Oxyz is expressed by the angle of attack  $\alpha \approx -(V_a)_z/(V_a)_x$ , and the side-slip angle  $\beta \approx (V_a)_y/(V_a)_x$ . The relation between  $V_g$  and Oxyz is expressed by the newly defined ascending angle

$\eta \approx (V_a)_z/(V_a)_x$ , and the lateral shift angle  $s \approx (V_a)_y/(V_a)_x$ . The wind is a function of time  $W(t)$ , and is given in the form of its three components in the inertial system  $O_g^x g^y g^z g$ , that is,  $W_{xg}(t), W_{yg}(t), W_{zg}(t)$ . The direction cosines between the moving system Oxyz and the inertial system  $O_g^x g^y g^z g$  is given in Table 1.

Table 1

Inertial System Moving System	$0_s x_s$	$0_s y_s$	$0_s z_s$
$0_x$	$\cos \vartheta \cos \psi$	$\sin \vartheta$	$-\cos \vartheta \sin \psi$
$0_y$	$\sin \vartheta \sin \psi - \cos \vartheta \cos \psi$	$\cos \vartheta$	$\sin \vartheta \cos \psi + \cos \vartheta \sin \psi$
$0_z$	$\sin \vartheta \sin \psi \cos \vartheta + \cos \vartheta \sin \psi$	$-\sin \vartheta \cos \vartheta$	$\cos \vartheta \cos \psi - \sin \vartheta \sin \psi$

If the rotational motion of the atmosphere is not considered, that is,  $\Omega = 0$ , and equation (1) is projected to  $Oxyz$ , the dynamic equations with the absolute motion parameters are obtained:

$$\left. \begin{aligned} m d(V_s)_x / dt &= m[\omega_x(V_s)_x - \omega_y(V_s)_y] + G_x + A_x + T_x \\ m d(V_s)_y / dt &= m[\omega_x(V_s)_y - \omega_z(V_s)_z] + G_y + A_y + T_y \\ m d(V_s)_z / dt &= m[\omega_y(V_s)_z - \omega_z(V_s)_y] + G_z + A_z + T_z \\ J_x \frac{d\omega_x}{dt} &= -(J_x - J_y)\omega_x\omega_y - J_{xy}(\omega_x\omega_z - d\omega_y/dt) + M_x + \dots \\ J_y \frac{d\omega_y}{dt} &= -(J_x - J_z)\omega_x\omega_z + J_{xz}(\omega_x\omega_y - d\omega_z/dt) + M_y + \dots \\ J_z \frac{d\omega_z}{dt} &= -(J_y - J_z)\omega_y\omega_z - J_{yz}(\omega_y^2 - \omega_z^2) + M_z + \dots \end{aligned} \right\}$$

(3.1)

The initial conditions are  $(V_s)_s|_{t=0} = (V_s)_s(0), \dots, \omega_s|_{t=0} = \omega_s(0), \dots$ .  
The kinematic equations are

$$\left. \begin{aligned} (V_s)_{s_x} &= (V_s)_x \cos \vartheta \cos \psi + (V_s)_y (\sin \vartheta \sin \psi - \cos \vartheta \sin \psi \cos \psi) \\ &\quad + (V_s)_z (\sin \vartheta \sin \psi \cos \psi + \cos \vartheta \sin \psi) \\ (V_s)_{s_y} &= (V_s)_x \sin \vartheta + (V_s)_y \cos \vartheta \cos \psi + (V_s)_z (-\sin \vartheta \cos \psi) \\ (V_s)_{s_z} &= (V_s)_x (-\cos \vartheta \sin \psi) + (V_s)_y (\sin \vartheta \cos \psi + \cos \vartheta \sin \psi \sin \psi) \\ &\quad + (V_s)_z (\cos \vartheta \cos \psi - \sin \vartheta \sin \psi \sin \psi) \end{aligned} \right\}$$

(3.2)

$$\begin{aligned}
\frac{d\phi}{dt} &= \frac{\omega_x \cos \gamma - \omega_z \sin \gamma}{\cos \gamma}, \quad \frac{d\theta}{dt} = \omega_z \sin \gamma + \omega_x \cos \gamma, \quad \frac{d\gamma}{dt} = \omega_x - \frac{d\phi}{dt} \sin \theta \\
(V_s)_x &= (V_s)_x - W_{xz}, \quad (V_s)_y = (V_s)_y - W_{yz}, \quad (V_s)_z = (V_s)_z - W_z \\
W_x &= W_{xz} \cos \theta \cos \phi + W_{yz} \sin \theta \cos \phi + W_{zz} \cos \theta \sin \phi \\
W_y &= W_{xz} (\sin \gamma \sin \phi - \cos \gamma \sin \theta \cos \phi) + W_{yz} \cos \gamma \cos \theta \\
W_z &= W_{xz} (\sin \gamma \sin \theta \cos \phi + \cos \gamma \sin \theta \sin \phi) - W_{yz} \sin \gamma \cos \theta \\
V_s &= \sqrt{(V_s)_x^2 + (V_s)_y^2 + (V_s)_z^2}, \quad \alpha \approx -(V_s)_z / (V_s)_x, \\
\beta &\approx (V_s)_x / (V_s)_z, \quad M = V_s / \alpha
\end{aligned} \tag{3.2}$$

Similarly, by setting  $\Omega = 0$  and projecting equations (2) to the Oxyz system, the dynamic equations with the relative motion parameters are obtained:

$$\begin{aligned}
m \frac{d(V_s)_x}{dt} &= G_x + A_x + T_x - m \frac{dW_x}{dt} + m[\omega_z(V_s)_y - \omega_x(V_s)_z] \\
m \frac{d(V_s)_y}{dt} &= G_y + A_y + T_y - m \frac{dW_y}{dt} + m[\omega_x(V_s)_z - \omega_z(V_s)_x] \\
m \frac{d(V_s)_z}{dt} &= G_z + A_z + T_z - m \frac{dW_z}{dt} + m[\omega_x(V_s)_y - \omega_y(V_s)_x] \\
J_x \frac{d\omega_x}{dt} &= -(J_x - J_z)\omega_y\omega_z - J_{xy} \left( \omega_x\omega_z - \frac{d\omega_x}{dt} \right) + M_x + \dots \\
J_y \frac{d\omega_y}{dt} &= -(J_x - J_z)\omega_x\omega_z + J_{xz} \left( \omega_y\omega_z - \frac{d\omega_y}{dt} \right) + M_y + \dots \\
J_z \frac{d\omega_z}{dt} &= -(J_y - J_x)\omega_x\omega_y - J_{xy} \left( \omega_x^2 - \omega_y^2 \right) + M_z + \dots
\end{aligned} \tag{4.1}$$

The initial conditions are  $(V_s)_x|_{t=0} = (V_s)_x(0), \dots, \omega_x|_{t=0} = \omega_x(0)$ .  
The kinematic equations are

$$\begin{aligned}
(V_s)_{xz} &= (V_s)_x \cos \theta \cos \phi + (V_s)_y (\sin \gamma \sin \phi - \cos \gamma \sin \theta \cos \phi) \\
&\quad + (V_s)_z (\sin \gamma \sin \theta \cos \phi + \cos \gamma \sin \phi) \\
(V_s)_{yz} &= (V_s)_x \sin \theta + (V_s)_y \cos \gamma \cos \theta + (V_s)_z (-\sin \gamma \cos \theta) \\
(V_s)_{zz} &= (V_s)_x (-\cos \theta \sin \phi) + (V_s)_y (\sin \gamma \cos \phi + \cos \gamma \sin \theta \sin \phi) \\
&\quad + (V_s)_z (\cos \gamma \cos \phi - \sin \gamma \sin \theta \sin \phi) \\
\frac{d\phi}{dt} &= \frac{\omega_x \cos \gamma - \omega_z \sin \gamma}{\cos \theta}, \quad \frac{d\theta}{dt} = \omega_z \sin \gamma - \omega_x \cos \gamma, \quad \frac{d\gamma}{dt} = \omega_x - \frac{d\phi}{dt} \sin \theta \\
V_s &= \sqrt{(V_s)_x^2 + (V_s)_y^2 + (V_s)_z^2}, \quad \alpha \approx -(V_s)_z / (V_s)_x, \\
\beta &\approx (V_s)_x / (V_s)_z, \quad M = V_s / \alpha \\
(V_s)_{xz} &= (V_s)_x + W_{xz}, \quad (V_s)_{yz} = (V_s)_y + W_{yz}, \quad (V_s)_{zz} = (V_s)_z + W_{zz}
\end{aligned} \tag{4.2}$$

Let  $X, Y, Z$  be the drag force, the lift force and the side force, and let  $\phi$  be the rigger's angle of incidence of the engine. Then the components of  $A, T, G$  in equations (3.1), (4.1) are equal. From the direction cosines relation between  $X, Y, Z, G, T$  and  $Oxyz$ , the following equations are obtained:

$$\begin{aligned} A_x &= -X \cos \alpha \cos \beta + Y \sin \alpha + Z \cos \alpha \sin \beta, \\ A_y &= X \sin \alpha \cos \beta + Y \cos \alpha - Z \sin \alpha \sin \beta, \\ A_z &= -X \sin \beta - Z \cos \beta, \\ G_x &= -G \sin \vartheta, \quad G_y = -G \cos \vartheta \cos \gamma, \quad G_z = G \cos \vartheta \sin \gamma, \\ T_x &= T \cos \varphi, \end{aligned}$$

Generally,  $T_y = T_z = 0$ . In engineering, the space motion is always simplified into the longitudinal and lateral motions. When studying the lateral motion, steady flight in the longitudinal direction is assumed, that is,  $(V_a)_x = (V_a)_{x_0} = \text{constant}$ ,  $\vartheta = \vartheta_0$ , and the elevation does not change. Simplifying equations (4.1) and (4.2) accordingly, the equations of motion in the lateral direction with relative motion parameters are obtained:

$$\left. \begin{aligned} m \frac{d(V_s)_z}{dt} &= G_z + A_z + T_z = m \frac{dW_z}{dt} + m\omega_z (V_s)_{z_0} \\ J_z \frac{d\omega_z}{dt} &= M_z + J_{zz} \frac{d\omega_y}{dt} + \dots \\ J_y \frac{d\omega_y}{dt} &= M_y - J_{zy} \frac{d\omega_z}{dt} + \dots \\ (V_s)_{z_0} &= (V_s)_{z_0} \cos \vartheta \cos \psi + (V_s)_z (\sin \gamma \sin \vartheta_0 \cos \psi + \cos \gamma \sin \psi) \\ (V_s)_{y_0} &= (V_s)_{z_0} (-\cos \vartheta \sin \psi) + (V_s)_z (\cos \gamma \cos \psi - \sin \gamma \sin \vartheta_0 \sin \psi) \\ (V_s)_{x_0} &= (V_s)_{z_0} + W_{x_0}, \quad (V_s)_{y_0} = (V_s)_{z_0} + W_{y_0} \\ \frac{d\psi}{dt} &= \frac{\omega_y \cos \gamma}{\cos \vartheta_0}, \quad \frac{d\gamma}{dt} = \omega_z - \frac{d\psi}{dt} \sin \vartheta_0 \\ V_s &= \sqrt{(V_s)_{z_0}^2 + (V_s)_z^2}, \quad \beta \approx (V_s)_z / (V_s)_{z_0}, \quad M = V_s/a \end{aligned} \right\}$$

(5)

The first equation in (5) can also be expressed as

$$m \frac{dp}{ds} = [G_s + A_s + T_s - m \frac{d\omega_s}{ds} + m\omega_s(V_s)_s] / (V_s)_s$$

Similar treatment for equations (3.1) and (3.2) gives the equation of motion in the lateral direction in terms of the parameters of absolute motion, and the lateral shift angle  $\epsilon$  is obtained. In general,  $\beta \neq s \neq \mu$ . Hence the drift angle should be calculated according to  $\phi = \psi - \mu$  or  $\phi \approx \psi - \epsilon$ .

Assuming that vehicle is flying in a fixed direction, that is,  $\psi = \psi_0 = 0$ , and after simplifying equations (4.1) and (4.2), the equations of motion in the longitudinal direction expressed in terms of the parameters of relative motion are

$$\left. \begin{aligned} m \frac{d(V_s)_s}{ds} &= G_s + A_s + T_s - m \frac{dW_s}{ds} + m\omega_s(V_s)_s, \\ m \frac{d(V_s)_s}{ds} &= G_s + A_s + T_s - m \frac{dW_s}{ds} - m\omega_s(V_s)_s, J_s \frac{d\omega_s}{ds} = M_s + \dots \\ (V_s)_s &= (V_s)_s \cos \vartheta - (V_s)_s \sin \vartheta, (V_s)_{s_s} = (V_s)_s \sin \vartheta + (V_s)_s \cos \vartheta \\ d\vartheta/ds &= \omega_s, V_s = \sqrt{(V_s)_s^2 + (V_s)_{s_s}^2}, a = -(V_s)_{s_s}/(V_s)_s, M = V_s/a \end{aligned} \right\} \quad (6)$$

The first equation in (6) can also be expressed as

$$m \frac{da}{ds} = [G_s + A_s + T_s - m \frac{dW_s}{ds} - m\omega_s(V_s)_s] / (V_s)_s$$

Same treatment for equations (3.1) and (3.2) gives the equations of motion in the lateral direction in terms of the parameters of absolute motion, and the ascending angle  $\eta$  is then obtained. In general,  $\eta \neq \alpha$ , and  $\theta$  can only be calculated according to  $\theta = \alpha - \eta$ . The initial conditions of equations (5) and (6) are the same as those of the original equations. The components of  $A$ ,  $G$ ,  $T$  should be simplified correspondingly.

The above discussion shows that the relative motion and the absolute motion of the equations of flight mechanics with wind should be distinguished. The corresponding initial conditions should be taken.

#### REFERENCES

- [1] Etkin, B., Dynamics of Flight (Stability and Control), John Wiley & Sons, Inc., New York (1959). 310—339.
- [2] Etkin, B., Dynamics of Atmospheric Flight, John Wiley & Sons, Inc., New York (1972). 529—563.
- [3] Gera, J., The influence of vertical wind gradients on the longitudinal motion of airplanes, NASA TN D-6430 (1971).
- [4] Franklin, J. A., Turbulence and lateral-directional flying qualities, NASA CR-1718 (1971).
- [5] Gorbatenko, S. A., et al. Flight Mechanics, Mashinostroyeniye, Moscow, 1969  
pp. 212-222
- [6] Chang, Po-su, et al., Chinese version of "Manual of Flight Mechanics", vol. 1, Defense Industry Publication Service, 1974, pp. 242-253.